# From nilpotent element to finite $W$-algebra

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# Outline

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1. Finite $W$-algebra of $\mathfrak{gl}_M$
2. Yangian associated to $\mathfrak{gl}_m$
3. Presentation of $W$-algebra in terms of Yangian
4. Super version of the story

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# Nilpotent matrix in Linear Algebra

- Consider $e \in M_M(\mathbb{C}) \cong \mathfrak{gl}_M(\mathbb{C})$.
- Recall that $e$ is called nilpotent if $e^k = 0$ for some $k$ large enough.
- **Question:** How many nilpotent matrices in $M_M(\mathbb{C})$ do we have, up to similarity (the number of orbits of nilpotent matrices under the $GL_M(\mathbb{C})$-conjugation)?

# Nilpotent matrix in Linear Algebra

- Base field is $\mathbb{C}$, so every matrix can be turned into its Jordan form.
- $e$ is nilpotent if and only if 0 is its only eigenvalue.
- We may arrange the Jordan blocks in a decreasing order with respect to their sizes.
- **Answer:** $\mathcal{P}(M)$, the partition function.
- $\mathcal{P}(M) =$ the number of ways to express $M$ as a sum of positive integers

  $=$ the number of partitions of $M$

  $=$ the number of Young diagrams with $M$ boxes

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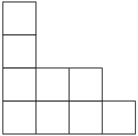
# Example

- Consider $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4, 3, 1, 1)$, a partition of 9.
- It corresponds to a nilpotent $9 \times 9$ matrix $e = J_4 \oplus J_3 \oplus J_1 \oplus J_1$, where $J_k$ is the Jordan block of size $k$ with eigenvalue 0. For example, $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $\lambda$ corresponds to the following Young diagram (in French style)



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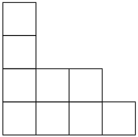

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# Finite $W$-algebra

- Let $e \in \mathfrak{g} = \mathfrak{gl}_M(\mathbb{C})$ be given. One can associated a very complicated algebra structure to this $e$, called (finite) $W$-algebra.
- This structure is hidden in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.
- We need many other things in order to define that structure.
- Recall the Lie bracket, or commutator notation

$$[x, y] := xy - yx, \quad \forall x, y \in \mathfrak{g}$$

- Take any $x \in \mathfrak{g}$. The derivation $ad_x : \mathfrak{g} \to \mathfrak{g}$ given by $ad_x(y) := [x, y]$ is a linear map.
- If $h \in \mathfrak{g}$ is semisimple ($=$ diagonalizable), then $\mathfrak{g}$ decomposes into a direct sum of eigenspaces of $ad_h$.

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# Good grading and good pair

- We say $(e, h)$ forms a good pair of $\mathfrak{g}$ if

  (i) $e \in \mathfrak{g}$ is nilpotent and $h \in \mathfrak{g}$ is semisimple.

  (ii) $\mathrm{ad}_h$ gives a good grading on $\mathfrak{g}$, which means that

    (a) eigenvalues(=gradings) of $ad_h$ are all integers:

    $$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j), \text{ where } \mathfrak{g}(j) := \{x \in \mathfrak{g} | [h, x] = jx\}$$

    (b) $e \in \mathfrak{g}(2)$ (that is, $[h, e] = 2e$).

    (c) $\mathrm{ad}_e : \mathfrak{g}(j) \to \mathfrak{g}(j+2)$ is injective for $j \leq -1$

    (d) $\mathrm{ad}_e : \mathfrak{g}(j) \to \mathfrak{g}(j+2)$ is surjective for $j \geq -1$

  (iii) In addition, if $\mathfrak{g}(j) = 0$ for all odd $j$, then we say $\mathrm{ad}_h$ gives an even good grading and $(e, h)$ form an even good pair.

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## Example (Dynkin Grading)

Take any nilpotent $e \in \mathfrak{g}$. Always exists an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{g}$ by Jacobson-Morozov Theorem. Then $(e, h)$ is a good pair by $\mathfrak{sl}_2$-repn theory.

- Let $h =$diag$(1, 0, -1)$, $e = e_{13}$. Then $f = e_{31}$ will produce an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{gl}_3$. The grading of $\mathfrak{gl}_3$ given by $\mathrm{ad}_h$ is given as below

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$$

  Easy to check that $(e, h)$ forms a good pair (but not even) for $\mathfrak{gl}_3$.

- Therefore, good pair always exists for any given $e$.

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For a fixed $e$, there exist other good pairs in general.

- Let $e = e_{13}$ (same as above) and $h' =$diag$(1, 1, -1)$. An easy calculation shows that $(e, h')$ also forms an even good pair for $\mathfrak{gl}_3$. The grading of $\mathfrak{gl}_3$ given by ad $h'$ is recored as

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- Fact: even good pair always exists for any given $e$.

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# Definition of finite $W$-algebra

- Given an even good pair $(e, h)$.
- Define the following subalgebras

$$\mathfrak{m} = \bigoplus_{j \leq -2} \mathfrak{g}(j) \,, \qquad \mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}(j).$$

- Define $\chi \in \mathfrak{g}^*$ by

$$\chi(y) := tr(y \cdot e), \ \forall y \in \mathfrak{g},$$

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- $I_\chi :=$ the left ideal of $U(\mathfrak{g})$ generated by $\{a - \chi(a) | a \in \mathfrak{m}\}$.
- PBW Theorem $\Rightarrow U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_\chi$ as vector spaces.
- $\mathrm{pr}_\chi : U(\mathfrak{g}) \to U(\mathfrak{p})$ the natural projection.
- $\overline{\mathrm{pr}}_\chi : U(\mathfrak{g})/I_\chi \to U(\mathfrak{p})$ isomorphism of vector space.
- Note that $U(\mathfrak{g})/I_\chi$ is a quotient algebra since we only quotient over a left ideal.

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# Definition of finite $W$-algebra

- Define a $\chi$-twisted adjoint action of $\mathfrak{m}$ on $U(\mathfrak{p}) \cong U(\mathfrak{g})/I_\chi$ by

$$a \cdot y := \mathrm{pr}_\chi([a, y]), \quad a \in \mathfrak{m}, y \in U(\mathfrak{p})$$

- An element $y \in U(\mathfrak{p})$ is annihilated by $a \in \mathfrak{m}$ means the following

$$a \cdot y = 0 \Leftrightarrow [a, y] \in I_\chi \Leftrightarrow ay - ya \in I_\chi \Leftrightarrow \big(a - \chi(a)\big)y \in I_\chi$$

since $a = \chi(a)$ in $I_\chi$.

- We collect all elements in $U(\mathfrak{p})$ that are annihilated by every elements of $\mathfrak{m}$, and they do form an algebra by the multiplication in $U(\mathfrak{p})$.

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# Definition of finite $W$-algebra

**Definition (Kostant, Lynch)**

The *finite W-algebra* $\mathcal{W}_{e,h}$ associated to $(e, h)$ is defined by

$$\begin{aligned}\mathcal{W}_{e,h} :=& U(\mathfrak{p})^{\operatorname{ad}\mathfrak{m}} \\ =& \{y \in U(\mathfrak{p}) | \big(a - \chi(a)\big) y \in I_\chi, \forall a \in \mathfrak{m}\}\end{aligned}$$

$=$ elements in $U(\mathfrak{p})$ that fall into $I_\chi$ under the $\chi$-twisted action of $\mathfrak{m}$

$=$ the set of Whittaker vectors in $U(\mathfrak{p}) \cong U(\mathfrak{g})/I_\chi$

- Trivial example: $e = h = 0$, then $\mathfrak{p} = \mathfrak{g}$ and $\mathfrak{m} = 0$. Hence $\mathcal{W}_{e,h} = U(\mathfrak{g})$.

- Our definition here is a simplified version. In general (the grading is good but not even), the definition is much more complicated. But never mind, up to isomorphism they are all the same to us!

**Theorem (Gan-Ginzburg'02, IMRN)**

*Up to isomorphism, the definition of W-algebra does not depend on the choice of the Lagrangian $\mathfrak{l}$ (appear for non-even good grading case).*

**Theorem (Brundan-Goodwin'07, Proc. LMS)**

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- There are other equivalent approaches to define $\mathcal{W}_{e,h}$, which means that it is an object in the intersection of different branches of mathematics. As a result, there are different approaches with different emphases to study it:
  - [Boer-Tjin: 93 CMP]
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  - [Premet: 95 Inv., 02 Adv., 07 J. EMS., 07 Mosc. Math. J., 10 Adv.]
- We roughly explain another definition here. The restriction of $\chi \in \mathfrak{g}^*$ to $\mathfrak{m}$ gives a 1-dim repn of $\mathfrak{m}$ and hence of $U(\mathfrak{m})$, denoted by $\mathbb{C}_\chi$.
- Consider the following induced representation of $U(\mathfrak{g})$ called the *generalized Gelfand-Graev representation*

$$\mathtt{Q}_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi$$

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- Take any $u \in \mathcal{W}_{e,h} = \{y \in U(\mathfrak{p}) \mid [a, y] = (a - \chi(a))y \in I_\chi \;\; \forall a \in \mathfrak{m}\}$
- Note that $I_\chi$ is invariant under right multiplication by $u$, which makes $\mathtt{Q}_\chi \cong U(\mathfrak{g})/I_\chi$ into a $(U(\mathfrak{g}), \mathcal{W}_{e,h})$-bimodule.
- One can show that the associated algebra homomorphism

$$\mathcal{W}_{e,h} \to \operatorname{End}_{U(\mathfrak{g})} \mathtt{Q}_\chi{}^{op} \qquad\qquad u \mapsto r_u$$

is actually an isomorphism. This gives an alternative definition of $W$-algebra as an endomorphism algebra.

**Definition (Premet)**

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Take $\mathfrak{sl}_2 = \{e = e_{12}, h = e_{11} - e_{22}, f = e_{21}\}$ in $\mathfrak{gl}_2$ and $(e, h)$ is a (Dynkin) good pair. The grading is recorded as $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Then

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In fact, this special example reveals some important facts.

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# Kostant's Theorem

- The study of $W$-algebra originated from Kostant's study on nilpotent orbits of $\mathfrak{g}$.

**Theorem (Kostant'78, Invent. Math.)**

*$\mathfrak{g} = \mathfrak{gl}_M$, $(e, h)$ a good pair with $e$ principal (also called regular) nilpotent. Then $\mathcal{W}_{e,h} \cong Z(\mathfrak{g})$.*

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- Recall $e = 0 \xleftrightarrow{\textit{Jordan type}} e = (1^M)$ (one column Young diagram)

  We have seen in this case the $W$-algebra is the largest one: $\mathcal{W}_{e,h} = U(\mathfrak{g})$.

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# Pyramid

- Observing the definition, one sees that $W$-algebra is determined by the good pair $(e, h)$.
- Pyramid: a convenient diagram simultaneously recording $e$ and $h$.
- Let $\lambda$ be a partition of $M$ = a Young diagram with $M$ boxes in French style (longest row in bottom, left justified).
- A pyramid is a diagram obtained by discretely shifting rows of the Young diagram $\lambda$ such that no bricks hanging in the air.
- Discretely means that the moving distance for each row is an integral multiple of the side length of a box.
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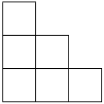
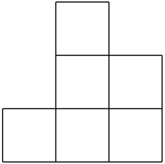
# Example of Pyramid

- For example, $\lambda = (3, 2, 1) =$  (it is itself a pyramid)



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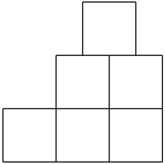

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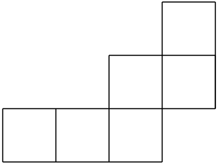

(X)

not discretely shift

# From $\pi$ to a good pair

$$\pi = \begin{array}{c|c|c|c}
 & & \boxed{2} & \\
 & & \boxed{3} & \boxed{5} \\
 & \boxed{1} & \boxed{4} & \boxed{6} \\
\hline
x\text{-coordinate} & -2 & 0 & 2
\end{array}$$

- $e(\pi) = e_{3,5} + e_{1,4} + e_{4,6} \in \mathfrak{gl}_6$.
- $h(\pi) = -\mathrm{diag}(-2, 0, 0, 0, 2, 2) \in \mathfrak{gl}_6$.
- Easily checked : $\big(e(\pi), h(\pi)\big)$ forms a good pair of $\mathfrak{gl}_6$.

# Every good pair comes from a Pyramid

- $e \longrightarrow \pi \longrightarrow (e(\pi), h(\pi))$. (existence of good pair for any $e$)
- In fact, the set of pyramids *classifies* all good pairs, due to the following theorem:

**Theorem (Elashvili-Kac'05)**

*Every good pair must come from a pyramid. That is, for any good pair $(e, h)$, there exists some pyramid $\pi$ such that $e = e(\pi)$ and $h = h(\pi)$.*
*Note: this results holds for non-even good pair as well*

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# Pyramid and shift matrix

- Equivalently, one can express a pyramid $\pi$ by a shift matrix

$$\sigma = (s_{i,j})_{1 \leq i,j \leq m}$$

$m$=height of $\pi$, $s_{ij} \in \mathbb{Z}_{\geq} 0$ satisfy the following condition

$$s_{i,j} + s_{j,k} = s_{i,k}, \tag{1.1}$$

whenever $|i-j|+|j-k|=|i-k|$,
together with a natural number $\ell > s_{m,1} + s_{1,m}$

- The condition (1.1) implies that the whole matrix $\sigma$ can be recovered if a row and a column is known.
- The process can be easily reverted, obtaining a unique $\pi$ from a given pair $\ell$ and $\sigma$.

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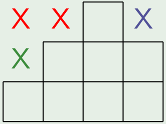
## Example

$$\ell = 4,\ \sigma = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \qquad \longrightarrow \qquad \pi =$$




- $\ell = 4$, $\sigma$ is $3 \times 3 \longrightarrow$ A rectangle $\Pi$ of base 4 height 3.
- Last row of $\sigma \longrightarrow$ Boxes to be removed in the left hand side of $\Pi$.
- Last column of $\sigma \longrightarrow$ Boxes to be removed in the right hand side of $\Pi$.
- Easy to construct the inverse map so we have a bijection.

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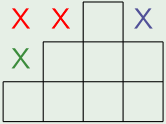


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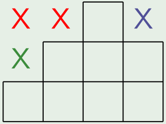


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# Summary about $\pi \longleftrightarrow (\sigma, \ell)$

$$\text{Width of } \pi \longleftrightarrow \ell$$

$$\text{Height of } \pi \longleftrightarrow \text{Size of } \sigma$$

$$\text{Shape of } \pi \longleftrightarrow \text{Entries of } \sigma$$

$$\text{Good pair } (e, h) \xleftrightarrow{1:1} \text{Pyramid } \pi \xleftrightarrow{1:1} \text{A matrix and integer } (\sigma, \ell)$$

# Outline

1. Finite $W$-algebra of $\mathfrak{gl}_M$
2. Yangian associated to $\mathfrak{gl}_m$
3. Presentation of $W$-algebra in terms of Yangian
4. Super version of the story

# Definition of $Y_m$

The Yangian associated to $\mathfrak{gl}_m$, denoted by $Y_m$, can be defined by several different but equivalent presentations. The first one we mention is

## Definition (RTT presentation)

$Y_m$: an associative algebra with generators

$$\left\{ t_{ij}^{(r)} \mid 1 \leq i, j \leq m; r \geq 0 \right\},$$

defining relations

$$t_{ij}^{(0)} := \delta_{ij},$$

$$[t_{ij}^{(r)}, t_{hk}^{(s)}] = \sum_{g=0}^{\min(r,s)-1} \left( t_{hj}^{(g)} t_{ik}^{(r+s-1-g)} - t_{hj}^{(r+s-1-g)} t_{ik}^{(g)} \right),$$

where the bracket stands for the commutator.

# Parabolic presentation for $Y_m$

For our purpose, we need a different presentation for $Y_m$ due to Brundan-Kleshchev.

## Theorem (BK'05, CMP)

*Let $\mu = (\mu_1, \mu_2, \dots, \mu_z)$ be a composition of $m$. $Y_m$ is isomorphic to the algebra generated by the following symbols*

$$\begin{aligned}
&\left\{ D_{a;i,j}^{(r)}, D_{a;i,j}^{\prime(r)} \,\middle|\, 1 \le a \le z,\ 1 \le i,j \le \mu_a,\ r \in \mathbb{Z}_{\ge 0} \right\}, \\
&\left\{ E_{b;h,k}^{(s)} \,\middle|\, 1 \le b < z,\ 1 \le h \le \mu_b, 1 \le k \le \mu_{b+1},\ s \in \mathbb{Z}_{\ge 1} \right\}, \\
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*subjected to certain relations.*

- These symbols are called the parabolic generators of $Y_m$, depending on $\mu$ by definition.

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# A few words about the presentation

- The Yangians (associated to reductive or semisimple Lie algebras) are firstly defined by Drinfeld around the 80's, in honor of C.N. Yang. (Yang-Baxter equation)
- Take $\mu = (m)$, we recover the RTT presentation.
- Take $\mu = (1^m)$, we recover an analogue of Drinfeld's presentation for $Y(\mathfrak{sl}_m)$.
- It is also proved in BK'05 that $Y_m$ is independent of the choice of $\mu$ up to isomorphism. We write $Y_\mu := Y_m$ to emphasize $\mu$ when necessary.

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# $W$-algebra and Yangian

- The connection between $W$-algebra and Yangian was firstly observed in [Ragoucy-Sorba'99, CMP] for special cases (rectangular pyramid). The general case (arbitrary $e$) is constructed by [BK'06].
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- To explain the result in [BK'06], we need to define the shifted Yangian, which is a subalgebra of $Y_m$.

# Shifted Yangian

- $(e, h) \longleftrightarrow \pi \longleftrightarrow (\sigma, \ell)$.
- Say $\sigma = (s_{i,j})$ is of size $m$.
- Take a composition $\mu = (\mu_1, \ldots, \mu_z)$ admissible to $\sigma$, which means that

$$s_{\mu_1+\mu_2+\cdots+\mu_{a-1}+i,\mu_1+\mu_2+\cdots+\mu_{a-1}+j} = 0$$

  for all $1 \le a \le z$, $1 \le i, j \le \mu_a$
- "admissible" means that square matrices of size $\mu_1, \mu_2, \ldots$ in the diagonal blocks of $\sigma$ are all zero matrices.

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# Shifted Yangian

For example, the following matrix is a shift matrix:

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 4 & 4 & 3 & 3 & 0 & 0 \\ 4 & 4 & 3 & 3 & 0 & 0 \end{pmatrix}$$

and $\mu = (1, 1, 2, 2)$ is minimal (the unique one with shortest length) admissible to $\sigma$.

$(1, 1, 2, 1, 1)$, $(1, 1, 1, 1, 1, 1)$ are also admissible.

# Shifted Yangian

- According to $\sigma$ and admissible $\mu$, we pick a subset of the parabolic generators we saw in the parabolic presentation of $Y_\mu$.
- Let $\mathcal{P}_{\mu,\sigma}$ be the union of following subsets in $Y_\mu$:

$$\{D_{a;i,j}^{(r)}, D_{a;i,j}'^{(r)} \,|\, 1 \le a \le z;\ 1 \le i,j \le \mu_a;\ r \ge 0\}$$
$$\{E_{b;h,k}^{(t)} \,|\, 1 \le b < z; 1 \le h \le \mu_b, 1 \le k \le \mu_{b+1}; t > s_{b,b+1}^{\mu}\}$$
$$\{F_{b;k,h}^{(t)} \,|\, 1 \le b < z;\ 1 \le h \le \mu_b, 1 \le k \le \mu_{b+1}; t > s_{b+1,b}^{\mu}\},$$

where $s_{a,b}^{\mu}$ means the entry in the $(a,b)$-th block of $\sigma$.
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(admissible condition $\Rightarrow$ all entries in the $(a,b)$-th block) are the same.

For example, take $\mu = (1,1,2)$ which is admissible to

$$\sigma = \begin{pmatrix} 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Then $\mathcal{P}_{\mu,\sigma}$ is the union

$$\{D_1^{(r)}, D_2^{(r)}, D_{3;i,j}^{(r)} \,|\, 1 \le i,j \le 2,\ r \ge 0\} \text{ (also those with } \prime\text{'s)},$$

$$\{E_1^{(s)} \,|\, s \ge 2\}, \quad \{E_{2;1,1}^{(s)}, E_{2;1,2}^{(s)} \,|\, s \ge 3\},$$

$$\{F_1^{(s)} \,|\, s \ge 1\}, \qquad \{F_{2;1,1}^{(s)}, F_{2;2,1}^{(s)} \,|\, s \ge 2\}.$$

# Shifted Yangian

- Define the shifted Yangian $Y_\mu(\sigma)$ to be the subalgebra of $Y_\mu$ generated by $\mathcal{P}_{\mu,\sigma}$.
- Not only the subalgebra, a presentation of $Y_\mu(\sigma)$ in terms of $\mathcal{P}_{\mu,\sigma}$ is also required to establish the connection to $W$-algebra, and it is also obtained in [BK'06].

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# Shifted Yangian

**Theorem (BK'06, Adv. Math.)**

1. *The shifted Yangian $Y_\mu(\sigma)$ is the abstract algebra generated by symbols in $\mathcal{P}_{\mu,\sigma}$ subjected to following relations.*
2. *One can identify $Y_\mu(\sigma)$ as a subalgebra of $Y_\mu$ by identifying the symbols in $\mathcal{P}_{\mu,\sigma}$ to corresponding elements in $Y_\mu = Y_m$ sharing the same name.*
3. *$Y_\mu(\sigma) = Y_\nu(\sigma)$ for any $\mu, \nu$ admissible to $\sigma$. (independent of the choices of the admissible shape)*

# Defining relations for shifted Yangian

$$
\begin{aligned}
D^{(0)}_{a;i,j} &= \delta_{ij}\,, \\
\sum_{t=0}^{r} D^{(t)}_{a;i,p} D'^{(r-t)}_{a;p,j} &= \delta_{r0}\delta_{ij}\,, \\
\left[D^{(r)}_{a;i,j}, D^{(s)}_{b;h,k}\right] &= \delta_{a,b} \sum_{t=0}^{min(r,s)-1} \left(D^{(t)}_{a;h,j} D^{(r+s-1-t)}_{a;i,k} - D^{(r+s-1-t)}_{a;h,j} D^{(t)}_{a;i,k}\right),
\end{aligned}
$$

# Relations(continued)

$$[D_{a;i,j}^{(r)}, E_{b;h,k}^{(s)}] = \delta_{a,b}\delta_{hj} \sum_{t=0}^{r-1} D_{a;i,p}^{(t)} E_{b;p,k}^{(r+s-1-t)} - \delta_{a,b+1} \sum_{t=0}^{r-1} D_{a;i,k}^{(t)} E_{b;h,j}^{(r+s-1-t)},$$

$$[D_{a;i,j}^{(r)}, F_{b;k,h}^{(s)}] = \delta_{a,b} \sum_{t=0}^{r-1} F_{b;k,p}^{(r+s-1-t)} D_{a;p,j}^{(t)} + \delta_{a,b+1} \sum_{t=0}^{r-1} F_{b;i,h}^{(r+s-1-t)} D_{a;k,j}^{(t)},$$

# Relations(continued)

$$[E_{a;i,j}^{(r)}, F_{b;k,h}^{(s)}] = \delta_{a,b}(-1) \sum_{t=0}^{r+s-1} D'^{(r+s-1-t)}_{a;i,h} D^{(t)}_{a+1;k,j},$$

$$[E_{a;i,j}^{(r)}, E_{a;h,k}^{(s)}] = \sum_{t=s^{\mu}_{a,a+1}+1}^{s-1} E^{(r+s-1-t)}_{a;i,k} E^{(t)}_{a;h,j} - \sum_{t=s^{\mu}_{a,a+1}+1}^{r-1} E^{(r+s-1-t)}_{a;i,k} E^{(t)}_{a;h,j},$$

$$[F_{a;i,j}^{(r)}, F_{a;h,k}^{(s)}] = \sum_{t=s^{\mu}_{a+1,a}+1}^{r-1} F^{(r+s-1-t)}_{a;i,k} F^{(t)}_{a;h,j} - \sum_{t=s^{\mu}_{a+1,a}+1}^{s-1} F^{(r+s-1-t)}_{a;i,k} F^{(t)}_{a;h,j},$$

# Relations(continued)

$$[E_{a;i,j}^{(r+1)}, E_{a+1;h,k}^{(s)}] - [E_{a;i,j}^{(r)}, E_{a+1;h,k}^{(s+1)}] = \delta_{h,j} E_{a;i,q}^{(r)} E_{a+1;q,k}^{(s)},$$

$$[F_{a;i,j}^{(r+1)}, F_{a+1;h,k}^{(s)}] - [F_{a;i,j}^{(r)}, F_{a+1;h,k}^{(s+1)}] = (-1)\delta_{i,k} F_{a+1;h,q}^{(s)} F_{a;q,j}^{(r)},$$

$$[E_{a;i,j}^{(r)}, E_{b;h,k}^{(s)}] = 0 \quad \text{if} \quad |b-a| > 1 \quad \text{or} \quad \text{if} \quad b = a+1 \text{ and } h \neq j,$$

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# Relations(continued)

$$[E_{a;i,j}^{(r)}, [E_{a;h,k}^{(s)}, E_{b;f,g}^{(t)}]] + [E_{a;i,j}^{(s)}, [E_{a;h,k}^{(r)}, E_{b;f,g}^{(t)}]] = 0 \quad \text{if } |a-b| \geq 1,$$

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In the relations, $p$ and $q$ are taken summed, and the indices $r$, $s$, $t$ run over all integers such that the corresponding elements make sense in $Y_\mu(\sigma)$.

## Remark

*Setting all $s_{i,j} = 0$ (i.e., $\sigma$ is the zero matrix), we get the defining relations of the whole Yangian $Y_\mu$ given in BK'05.*

# Truncation

- Recall that $\pi \leftrightarrow (\sigma, \ell)$. We have used $\sigma$ to define a subalgebra $Y_\mu(\sigma)$. Now it's time to get $\ell$ involved.
- Denote by $I_\ell$ the 2-sided ideal of $Y_\mu(\sigma)$ generated by

$$\{D_{1;i,j}^{(r)} \,|\, 1 \leq i,j \leq \mu_1,\ r \geq p_1\},$$

where $p_1$ is the number of boxes in the top row of $\pi$, which is determined by $\sigma$ and $\ell$.
- $Y_\mu^\ell(\sigma) := Y_\mu(\sigma)/I_\ell$, called the *shifted Yangian of level $\ell$*.

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# Main result of BK and consequences

**Theorem (BK'06, Adv. Math.)**

*There exists an algebra isomorphism between $Y_\mu^\ell(\sigma)$ and $\mathcal{W}_\pi$.*

- One may study representation theory of $W$-algebra via tools from representation theory of Yangian, which people understand better (methods developed by Drinfeld, Molev, Nazarov, Olshanski, Tarasov,...etc). [BK'08, Mem. AMS.][BK'08, Sel. Math.]
- Triggered the study of shifted Yangian in geometric representation theory (a series of papers about Coulomb branches by different groups of people: Braverman, Finkelberg, Kamnitzer, Kodera, Nakajima, Webster, Weeks, Yacobi, etc.).

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# $W$-superalgebra

- One defines (finite) $W$-superalgebra from a good pair $(e, h)$ in $\mathfrak{gl}_{M|N}$ in a very similar way to the $\mathfrak{gl}_M$ case with some modifications:

  1. Choose $e \in (\mathfrak{gl}_{M|N})_{\bar{0}}$ to be even nilpotent (assume this from now on)
  2. Replace tr by str when defining $\chi : \mathfrak{g} \to \mathbb{C}$, $\chi(y) = \mathrm{str}(y \cdot e)$.
  3. Same definition of $\mathfrak{m}, \mathfrak{p}$ and $\chi$-twisted action of $\mathfrak{m}$ on $\mathfrak{p}$, and collect the annihilated elements

$$\mathcal{W}_{e,h} := U(\mathfrak{p})^{ad\,\mathfrak{m}} = \{y \in U(\mathfrak{p}) | \big(a - \chi(a)\big)y \in I_\chi, \forall a \in \mathfrak{m}\}$$

- One can also describe a good pair $(e, h)$ by a pyramid, with some modification.

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- The Jordan type of $e$ is now described by a pair of partitions (of $M$ and $N$ respectively)

$$e \in \mathfrak{g}_{\bar{0}} \Rightarrow e = e_M \oplus e_N \quad \Big( (\mathfrak{gl}_{M|N})_{\bar{0}} \cong \mathfrak{gl}_M \oplus \mathfrak{gl}_N \Big)$$

- Stack the Young diagrams of $e_M$ and $e_N$ together, rearranging the rows if necessary, to form a new Young diagram.
- But we need to keep track of the origin of each row (from $e_M$ or $e_N$) by coloring the bricks, and then discretely shift the rows to form a pyramid which consists of two different colors of boxes.

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$$e \in \mathfrak{g}_{\bar{0}} \Rightarrow e = e_M \oplus e_N \quad \Big( (\mathfrak{gl}_{M|N})_{\bar{0}} \cong \mathfrak{gl}_M \oplus \mathfrak{gl}_N \Big)$$

- Stack the Young diagrams of $e_M$ and $e_N$ together, rearranging the rows if necessary, to form a new Young diagram.
- But we need to keep track of the origin of each row (from $e_M$ or $e_N$) by coloring the bricks, and then discretely shift the rows to form a pyramid which consists of two different colors of boxes.

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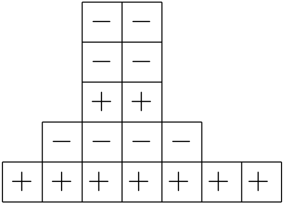
# Example

The following is a pyramid representing a good pair $(e, h)$ in $\mathfrak{gl}_{9|8}$:



- $e = e^+ + e^-$, where $e^+ \in \mathfrak{gl}_{9|0}$ of Jordan type (7,2) and $e^- \in \mathfrak{gl}_{0|8}$ of Jordan type (4,2,2).
- $e \in \mathfrak{g}_{\bar{0}} \Rightarrow$ every bricks in the same row have the same color ($+$ or $-$).
- The $\pm$-labeling can be recorded as $- - + - +$ from top to bottom, or 11010 ($+ \leftrightarrow 0$, $- \leftrightarrow 1$). This is called a 01-sequence.

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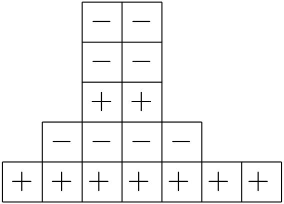


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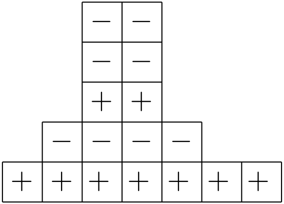

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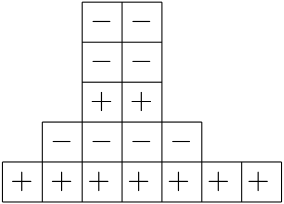


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# Pyramids and 01-sequence

- Labeling $+$ boxes with $1, \cdots, M$ and $-$ boxes with $\overline{1}, \cdots, \overline{N}$, one can explicitly write down matrices $e_\pi$ and $h_\pi$ in $\mathfrak{gl}_{M|N}$ exactly the same way as before, and they do form a good pair.

**Theorem (Hoyt'12)**

*Good pairs $(e, h)$ in $\mathfrak{gl}_{M|N}$ are classified by colored pyramids: pyramids with two different colors ($+$ and $-$) of bricks.*

- It makes sense to write $\mathcal{W}_\pi = \mathcal{W}_{(e_\pi, h_\pi)}$ and every $W$-superalgebra must be of this form due the the theorem above.

- The pyramid $\pi$ now can be uniquely recorded by a triple $(\sigma, \ell, \mathfrak{b})$ where $\sigma$ is a shift matrix of size $m + n$, $\ell$ is the number of boxes in the bottom row of $\pi$, and $\mathfrak{b}$ is the 01-sequence recording the color of rows from top to bottom.

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# Example

Take the standard basis for superspace with superdimension $(9|8)$.

$$\pi = \begin{array}{ccccccc} & & \bar{2} & \bar{5} & & & \\ & & \bar{3} & \bar{6} & & & \\ & & 3 & 5 & & & \\ & \bar{1} & \bar{4} & \bar{7} & \bar{8} & & \\ 1 & 2 & 4 & 6 & 7 & 8 & 9 \end{array}$$

- $e_\pi = e^+ + e^- \in (\mathfrak{gl}_{9|8})_{\bar{0}}$, where
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# Example(continued)

The corresponding shift matrix is given by

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}$$

- $\mathfrak{b} = 11010$ is the corresponding 01-sequence.
- $\mu = (3, 1, 1)$ is the minimal admissible shape.
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- Let $\mathfrak{b}$ be a fixed $0^m1^n$-sequence and let $\mu = (\mu_1, \ldots, \mu_z)$ be a composition of $m + n$.
- Decompose $\mathfrak{b}$ into $z$ subsequence $\mathfrak{b}_1\mathfrak{b}_2 \ldots \mathfrak{b}_z$ in the obvious way.
- For example, $\mu = (3, 1, 1)$ and $\mathfrak{b} = 11001$ in $Y_{2|3}$

$$\mathfrak{b} = \overbrace{110}^{\mathfrak{b}_1}\overbrace{0}^{\mathfrak{b}_2}\overbrace{1}^{\mathfrak{b}_3}.$$

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# Parabolic presentation of $Y_{m|n}$

## Theorem (P'16, CMP)

*Let $\mu = (\mu_1, \ldots, \mu_z)$ be any composition of $m+n$ and $\mathfrak{b}$ be any $0^m 1^n$-sequence. Define $Y_{\mu,\mathfrak{b}}$, or simply $Y_\mu$ for short, to be the abstract superalgebra generated by the following symbols*

$$\{D_{a;i,j}^{(r)}, D_{a;i,j}^{\prime(r)} \,|\, 1 \le a \le z, 1 \le i,j \le \mu_a, r \in \mathbb{Z}_{\ge 0}\},$$
$$\{E_{b;h,k}^{(t)} \,|\, 1 \le b < z, 1 \le h \le \mu_a, 1 \le k \le \mu_{a+1}, t \in \mathbb{Z}_{\ge 1}\},$$
$$\{F_{b;k,h}^{(t)} \,|\, 1 \le b < z, 1 \le h \le \mu_a, 1 \le k \le \mu_{a+1}, t \in \mathbb{Z}_{\ge 1}\},$$

*subject to certain relations (depending on $\mu$ and $\mathfrak{b}$). Then we have $Y_\mu \cong Y_{m|n}$ as a superalgebra. Moreover, the definition is independent of the choices of $\mu$ and $\mathfrak{b}$ up to isomorphism.*

# Parabolic presentation of $Y_{m|n}$

- We should explain how to define the parities of the generators.
- The parity of $D_{a;i,j}^{(r)}$, $r > 0$ is defined by

$$|D_{a;i,j}^{(r)}| := |i|_a + |j|_a (\mathrm{mod} 2)$$

- Similarly we set

$$|E_{b;h,k}^{(t)}| := |h|_b + |k|_{b+1} (\mathrm{mod} 2) \quad \text{and} \quad |F_{b;k,h}^{(t)}| := |k|_{b+1} + |h|_b (\mathrm{mod} 2)$$

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# Defining Relations for $Y_\mu = Y_{m|n}$

$$
\begin{aligned}
D^{(0)}_{a;i,j} &= \delta_{ij}\,, \\
\sum_{t=0}^{r} D^{(t)}_{a;i,p} D'^{(r-t)}_{a;p,j} &= \delta_{r0}\delta_{ij}\,, \\
\left[D^{(r)}_{a;i,j}, D^{(s)}_{b;h,k}\right] &= \delta_{ab}(-1)^{|i|_a|j|_a+|i|_a|h|_a+|j|_a|h|_a} \times \\
&\quad \sum_{t=0}^{min(r,s)-1} \left(D^{(t)}_{a;h,j} D^{(r+s-1-t)}_{a;i,k} - D^{(r+s-1-t)}_{a;h,j} D^{(t)}_{a;i,k}\right),
\end{aligned}
$$

# Defining Relations for $Y_\mu = Y_{m|n}$(continued)

$$[D^{(r)}_{a;i,j}, E^{(s)}_{b;h,k}] = \delta_{a,b}\delta_{hj}(-1)^{|h|_a|j|_a}\sum_{t=0}^{r-1} D^{(t)}_{a;i,p}E^{(r+s-1-t)}_{b;p,k}$$

$$- \delta_{a,b+1}(-1)^{|h|_b|k|_a+|h|_b|j|_a+|j|_a|k|_a}\sum_{t=0}^{r-1} D^{(t)}_{a;i,k}E^{(r+s-1-t)}_{b;h,j},$$

$$[D^{(r)}_{a;i,j}, F^{(s)}_{b;h,k}] = \delta_{a,b}(-1)^{|i|_a|j|_a+|h|_{a+1}|i|_a+|h|_{a+1}|j|_a}\sum_{t=0}^{r-1} F^{(r+s-1-t)}_{b;h,p}D^{(t)}_{a;p,j}$$

$$+ \delta_{a,b+1}(-1)^{|h|_a|k|_b+|h|_a|j|_a+|j|_a|k|_b}\sum_{t=0}^{r-1} F^{(r+s-1-t)}_{b;i,k}D^{(t)}_{a;h,j},$$

# Defining Relations for $Y_\mu = Y_{m|n}$(continued)

$$[E_{a;i,j}^{(r)}, F_{b;h,k}^{(s)}] = \delta_{a,b}(-1)^{|h|_{a+1}|k|_a+|j|_{a+1}|k|_a+|h|_{a+1}|j|_{a+1}+1} \sum_{t=0}^{r+s-1} D_{a;i,k}^{\prime(r+s-1-t)} D_{a+1;h,j}^{(t)},$$

$$[E_{a;i,j}^{(r)}, E_{a;h,k}^{(s)}] = (-1)^{|h|_a|j|_{a+1}+|j|_{a+1}|k|_{a+1}+|h|_a|k|_{a+1}} \times \Big(\sum_{t=1}^{s-1} E_{a;i,k}^{(r+s-1-t)} E_{a;h,j}^{(t)} - \sum_{t=1}^{r-1} E_{a;i,k}^{(r+s-1-t)} E_{a;h,j}^{(t)}\Big),$$

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## Defining Relations for $Y_\mu = Y_{m|n}$(continued)

$$[E_{a;i,j}^{(r+1)}, E_{a+1;h,k}^{(s)}] - [E_{a;i,j}^{(r)}, E_{a+1;h,k}^{(s+1)}] = (-1)^{|j|_{a+1}|h|_{a+1}} \delta_{h,j} E_{a;i,q}^{(r)} E_{a+1;q,k}^{(s)},$$

$$[F_{a;i,j}^{(r+1)}, F_{a+1;h,k}^{(s)}] - [F_{a;i,j}^{(r)}, F_{a+1;h,k}^{(s+1)}] = (-1)^{|i|_{a+1}(|j|_a+|h|_{a+2})+|j|_a|h|_{a+2}+1} \delta_{i,k} F_{a+1;h,q}^{(s)} F_{a;q,j}^{(r)},$$

$$[E_{a;i,j}^{(r)}, E_{b;h,k}^{(s)}] = 0 \qquad \text{if} \quad |b-a|>1 \quad \text{or} \quad \text{if} \;\; b=a+1 \text{ and } h \neq j,$$

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$$\big[E^{(r)}_{a;i,j}, [E^{(s)}_{a;h,k}, E^{(t)}_{b;f,g}]\big] + \big[E^{(s)}_{a;i,j}, [E^{(r)}_{a;h,k}, E^{(t)}_{b;f,g}]\big] = 0 \quad \text{if } |a-b| \geq 1,$$

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Last two: appear when $z \geq 4$, super phenomenon, didn't appear in [BK'05]. These come from similar relations given in [Gow'07, CMP], which are the special case when $\mu = (1^{m+n})$ and standard $\mathfrak{b}$ in our setting.

# Shifted super Yangian $Y_{m|n}(\sigma)$

- Following the same logic in the classical case, we use the parabolic presentation to define the subalgebra we need.
- Recall $\pi \longleftrightarrow (\sigma, \ell, \mathfrak{b})$.
- Choose a composition $\mu$ admissible to $\sigma$, we can define shifted super Yangian $Y_\mu(\sigma)$ exactly the same way as before: take a subset of the parabolic generators and use them to generate a subalgebra.
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# Main result 1

Set $\mathcal{P}_{\mu,\sigma}$ to be the union of the symbols
$\{D^{(r)}_{a;i,j}, D'^{(r)}_{a;i,j} \,|\, 1 \le a \le z;\ 1 \le i,j \le \mu_a;\ r \ge 0\}$,
$\{E^{(t)}_{b;h,k} \,|\, 1 \le b < z; 1 \le h \le \mu_b, 1 \le k \le \mu_{b+1}; t > s^{\mu}_{b,b+1}\}$,
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**Theorem (P'21, Adv. Math.)**

*The shifted super Yangian $Y_\mu(\sigma)$ is the abstract superalgebra generated by $\mathcal{P}_{\mu,\sigma}$ subjected to a set of defining relations.*

- *One can identify $Y_\mu(\sigma)$ as a subalgebra of $Y_\mu = Y_{m|n}$ by identifying the symbols in $\mathcal{P}_{\mu,\sigma}$ to corresponding elements in $Y_\mu$ sharing the same name (obtained via quasideterminant).*
- *Furthermore, $Y_\mu(\sigma) = Y_\nu(\sigma)$ for any $\mu, \nu$ admissible to $\sigma$.*

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- The defining relations are obtained by modifying the relations in [P'16] so that they make sense in $Y_\mu(\sigma)$.
- The most difficult part is to show the "extra" relations hold in $Y_\mu$:

$$\big[[E^{(r)}_{a;i,f_1}, E^{(t)}_{a+1;f_2,j}], [E^{(t)}_{a+1;h,g_1}, E^{(s)}_{a+2;g_2,k}]\big] = 0 \text{ when } |h|_{a+1} + |j|_{a+2} = 1,$$

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for all $t > s^\mu_{a+1,a+2}$ and $t > s^\mu_{a+2,a+1}$.

- The proof in [P'16] only works for $t = 1$.
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# Main result 2

- Similarly, define $I_\ell$ to be the 2-sided ideal of $Y_\mu(\sigma)$ generated by

$$\{D_{1;i,j}^{(r)} \mid 1 \leq i,j \leq \mu_1,\ r \geq p_1\},$$

and let $Y_\mu^\ell(\sigma)$ denote the quotient $Y_\mu(\sigma)/I_\ell$.

**Theorem (P'21, Adv. Math.)**

*There exists a superalgebra isomorphism between $Y_\mu^\ell(\sigma)$ and $\mathcal{W}_\pi$.*

- Note that the definition of $\mathcal{W}_{e,h}$ is again independent of the choices of the good grading, due to [Zhao'14]. Hence our results apply to all $W$-superalgebras (up to isomorphism).

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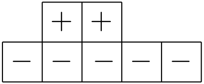
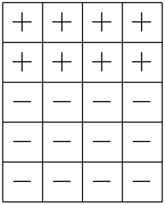
# Some earlier results in super case



$\pi$ : *principal*

Brown-Brundan-Goodwin'13

No extra relations!



$\pi$ : *rectangular*

Briot-Ragoucy'03

RTT works

# Our New Result

$$\begin{array}{|c|c|c|c|c|c|c|}
 & & - & - & & & \\
 & & + & + & & & \\
 & & - & - & & & \\
 & & + & + & + & & \\
 & - & - & - & - & & \\
+ & + & + & + & + & + & +
\end{array}$$

$\pi$: corresponds to an arbitrary even nilpotent element

P'21, Adv. Math.

**Thank you for your attention.**